Retrospectives on My Studies of Solid Mechanics (II)

- a new energy method based on the stationary principle of total energy –

By Tadahiko Kawai [+] and Etsu Kazama [++]

ABSTRACT

A new energy method is proposed in this article basing on the stationary principle of total energy introduced in the previous article.

As the first example of verification studies in-plane bending of a cantilever plate due to vertical tip shear is analyzed by the new energy method proposed:

1. A simple, but elaborate solution on the in-plane bending of a cantilever plate

This is one of fundamental problems in the strength of materials for structural engineers and professor S.P. Timoshenko gave a very simple and yet elaborate solution to this problem in his cerebrated textbook on the theory of elasticity [(1), (2)].

His solution is summarized as follows:

（1） governing equilibrium equation:

$$\left.\begin{aligned} \nabla^2 u + \frac{(1+\nu)}{2}\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + f_x = 0 \\ \nabla^2 v + \frac{(1+\nu)}{2}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right) + f_y = 0 \end{aligned}\right\}\ldots\ldots(1)$$

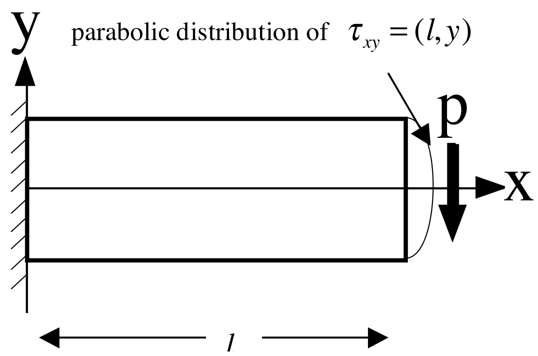


Fig.1 In-plane bending of a cantilever due to a boundary shear P

where (u (x,y), v (x,y)) is the unknown displacement vector of a cantilever plate and (f_x, f_y) are non-dimensional intensities of the body force vector (X,Y): $f_x = \frac{(1-\nu^2)X}{E}, f_y = \frac{(1-\nu^2)Y}{E}$,

E: Young modulus, ν : Poisson's ratio

the associated boundary conditions are given as follows:

$$\left.\begin{array}{llll} x=0 & : & clamped \quad edge & u(0,y)=0, v(0,y)=0 \\ y=\pm c & : & free \quad edge & \tau_{xy}(x,\pm c)=0, \sigma_y(x,\pm c)=0 \\ x=l & : & loading \quad edge & \sigma_y(l,y)=0 \\ & & & \tau_{xy} = -\dfrac{P}{2I}(c^2-y^2) \end{array}\right\}\ldots\ldots\ldots\ldots(2)$$

+) Professor, Emeritus, the University of Tokyo

++) Professor, Nagano College of Technology

To solve this problem, he replaced the clamped edge condition: u (0,y)= ν (0,y)=0
by

$$u(0,0) = \nu(0,0) = \frac{\partial u}{\partial y}(0,0) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

This is, the plate is fixed only at the origin.

Professor Timoshenko assumed first the following stress field of a given cantilever plate:

$$\sigma_x = \frac{P_{xy}}{I}, \sigma_y = 0, \tau_{xy} = -\frac{P}{2I}(c^2 - y^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

and integrating the associated strains, he derived the following solution for displacement field of a given cantilever plate:

$$\left.\begin{aligned} u(x,y) &= -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} - \frac{Py^3}{6GI} + \frac{P}{2EI}\left\{l^2 - 2(1+\nu)c^2\right\}y \\ \nu(x,y) &= \frac{\nu Pxy^2}{2EI} + \frac{\nu Px^3}{6EI} - \frac{Pl^3x}{2EI} + \frac{Pl^3}{3EI} \end{aligned}\right\} \quad \ldots\ldots..(5)$$

where $I = \frac{2}{3}tc^3$ is the moment of inertial about z axis, t is the plate thickness.

This solution represents the bending behavior of a cantilever beam with the effect of shear deformation so nicely that it has been accepted to use as a standard solution for bench mark testing of in-plane finite elements. However, it should be emphasized hat eq (5) is the exact closed form solution under the simplified clamped edge condition eq (3), but it is only one of approximate solutions of the resent problem.

2. Is convergency of CST element really poor in the bending analysis of a cantilever?

It is generally recognized since 40 years ago that the Constant Strain Triangular element is the standard finite element for analysis of plain elasticity problems, but it exhibits very poor convergency to the solution given by professor Timoshenko as shown in Fig.2 [(3),(4),(5)].

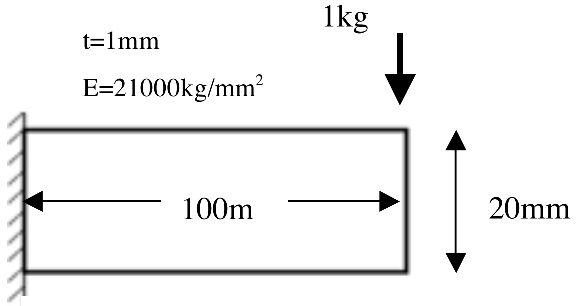
I could hardly believe such poor convergency as shown in Fig.2 of the CST element which gave usually satisfactory results for our general plane stress analysis.

In my early days of the FEM development, I attempted to clarify the reason why in vain.

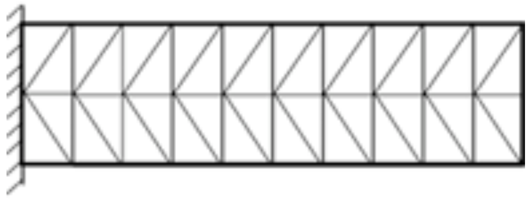
About 30 years passed without substantial progress in my research on this problem and since a few years ago I challenged one more to obtain the accurate solution using Energy Method with the following displacement function.

$$\left.\begin{aligned} u(x,y) &= x\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}x^m y^n \\ \nu(x,y) &= x\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{mn}x^m y^n \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(6)$$

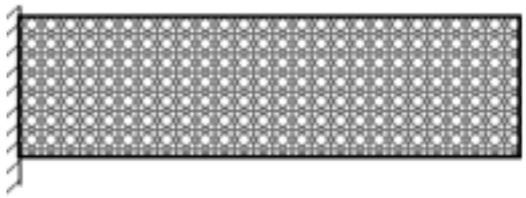
Eqs (6) satisfies completely the clamped edge conditions given as u (0,y)= ν (0,y)=0:



(i) dimensions and material constants

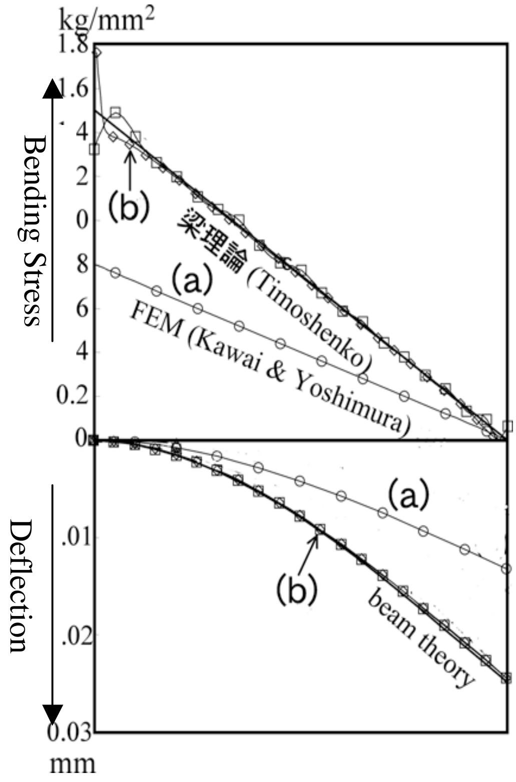


(a) 40 elements



(b) 2000 elements

(ii) mesh divisions used in FEM analysis



(iii) comparison of calculated deformation and bending stress

Fig. 2 Discrepancy of the FEM solution using CST elements and Timoshenko's solution on the in-plane bending of a cantilever plate. (aspect ratio=5)

I expected to obtain the better approximate solution easily well known Rayleigh-Ritz's procedure. Unfortunately, however, I could not derive a reliable solution by this method wasting much time again. Because the approximate solution oscillates no matter how many times may be taken even in the displacement computation. Quite recently, I have been convinced that it might be due to existence of stress singularities at both corners of the clamped edge. This suggests the rigorous clamped edge condition u (0,y)= ν (0,y)=0 is too difficult to satisfy.

Therefore, it is not generally possible to derive the exact solution because stresses become unbound at corners of the clamped edge even in the elastic range of the deformation and therefore one should be satisfied with the approximate solution. As a result of this study, it can be concluded that Timoshenko's solution is important from the viewpoint of beam theory, but it is not adequate to use it as the standard solution for the bench mark testing of in-plane finite elements if the aspect ratio l/2c is less than 5.

3. My challenge on the restoration of the FORCE METHOD

So I was obliged to find a rigorous of this problem for bench mark testing of plain finite elements, and I changed my solution procedure to the stress function approach.

This problem was analyzed recently using the unified energy method explained the previous article. Finite element analyses were conducted using the following non-equilibrium displacement function, NDOF of which is 16 as follows:

$$\left.\begin{aligned} u(x,y) &= u_0 - \chi_0 y + \varepsilon_{x0} x + \frac{1}{2}\gamma_{xy0} + a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x^2 y + a_5 xy^2 \\ v(x,y) &= v_0 - \chi_0 y + \varepsilon_{y0} y + \frac{1}{2}\gamma_{xy0} + b_1 x^2 + b_2 xy + b_3 y^2 + b_4 x^2 y + b_5 xy^2 \end{aligned}\right\} \quad \ldots\ldots\ldots(7)$$

when (u_0, v_0, χ_0) is the rigid body displacement vector of a given element, and $\left(\varepsilon_{xo}, \varepsilon_{y0}, \gamma_{xy}\right)$ is the constant strain vector of the same element.

For the equilibrium displacement functions of same NDOF is also derived using 4th order polynomial of z for $\varphi(z)$ and $\chi(z)$ of the following Goursat's stress function. In brief

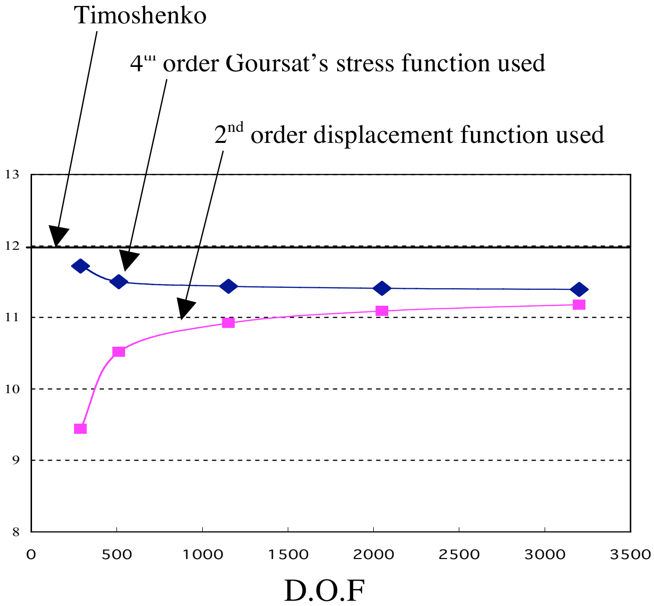
Airy's stress function F(x,y) = $\mathrm{Re}\left[\bar{z}\varphi(z) + \chi(z)\right]$ ……………………………(8)

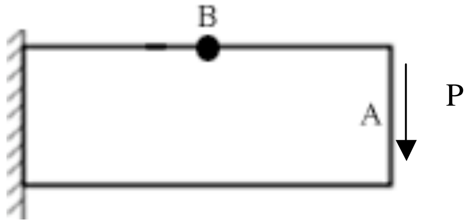
$$\varphi(z) = \sum_n^\infty A_n z^n, \chi(z) = \sum_n^\infty B_n z^n$$

$$\left.\begin{aligned} & z = x + iy, A_n = a_n + ib_n, B = c_n + id_n \\ & \sigma_x + \sigma_y = 4\,\mathrm{Re}[\varphi'(z)] \\ where \quad & \sigma_x - \sigma_y + 2i\tau_{xy} = 2[\bar{z}\varphi''(z) + \chi''(z)] \\ & 2G(u + iv) = \left(\frac{3-\nu}{1+\nu}\right)\varphi(z) - z\overline{\varphi'(z)} - \overline{\chi'(z)} \end{aligned}\right\}$$

A solution obtained using nonequilibrium displacement function is shown by the curve -■-, while the other solution using the equilibrium displacement is shown the curve -◆- in this figure. Fig.3 show the convergency of the calculated displacement ν_A and stress ρ_B respectively. It can be seen the curves -■- gives always the upper bound solution for both ν_A and ρ_B, on the other hand, the curve -◆- gives the lower bound solution clearly.

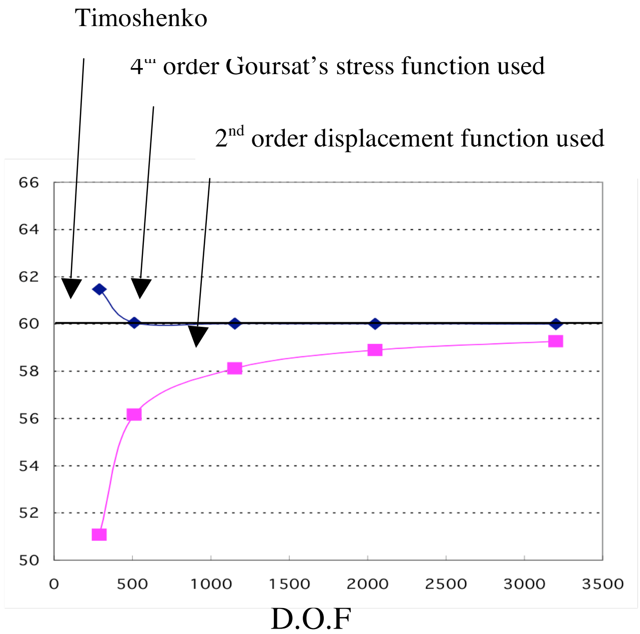
v_A: vertical displacement at the point A

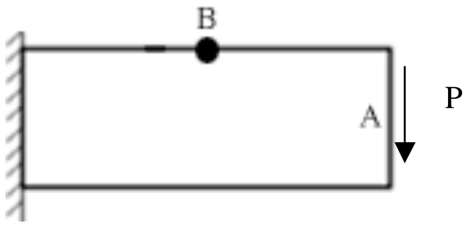




Mesh Div.x NDOF	stress function used	displacement function used
4x2x16	61.4766	51.0777
8x4x16	60.0641	56.1607
12x6x16	60.0287	58.1254
16x8x16	60.0138	58.8946
20x10x16	60.0071	59.2698

$(\sigma_X)_B$: stress at the point B





Mesh Div. x NDOF	stress function used	displacement function used
4x2x16	117195	9.4399
8x4x16	11.4996	10.5163
12x6x16	11.4347	10.9196
16x8x16	11.4063	11.0912
20x10x16	11.3909	11.178

Figure 3: In-plane bending analysis of a cantilever plate subjected to a boundary shear of parabolic distribution (divided by square mesh)

6. Conclusions

(1) The elastic bending of a cantilever does not exist because of the singularities at the both corners of the clamped edge. Therefore one should be satisfied with the approximate solution. (See Fig.2)

(2) Restoration of the FORCE METHOD was demonstrated using the stress function approach on the in-plane bending of a cantilever plate due to a boundary shear.

（３） The following variational theorems are confirmed by analysis of this cantilever bending problem.

(i) Displacement Method gives the upper bound solution for the stresses, hence the lower bound solution for the displacement.

(ii) Force Method gives the lower bound solution for the stresses, hence the upper bound solution for the displacement.

(iii) Therefore, the true displacement solutions must lei inbetween the DM and FM solutions.

（４） The elaborate solution for the cantilever problem by professor Timoshenko is indeed correct from the viewpoint of beam theory, but is not correct from the point of in-plane bending of a cantilever plate. Therefore it can be concluded that Timoshenko's solution is not adequate to use the standard solution for bench mark testing of in-plane FEM elements within the aspect ratio less than 5.

Acknowledgement

The author would like to express heartful thanks to Mr. Masahiko Kohnosu, manager of the ACT Co., Ltd and Mr. Mitsunari Kitayama of the Nagano Collage for their efforts showed in the course of the present paper preparation.

References

1. S. P. Timoshenko and J. N. Goodier, Theory of Elasticity, THIRD EDITION, McGraw-Hill; INTERNATIONAL BOOK COMPANY, 1982
2. Kyuichiro Washizu, Variational Methods in Elasticity and Plasticity, Pergamon Press: New York, 1965
3. R. H. Gallagher, Finite Element Analysis Fundamentals, Prentice-Hall, Inc., Englewood Cliffs, N. j. 1975
4. T. Kawai, The Force Method Revisited, Int. J. Num. Mech. Engng., 47, 275-286, (2000)
5. T. Kawai, Development of a Nodeless and consistent Finite Element Method - force method forever -, Proc. of the Fifth World Congress of Computational Mechanics, July 7-12, 2002 Vienna, Austria